EXAM 1

1. a) 2pt

$$V=\int_0^{\sqrt[3]{x}} 2\pi x\cdot x\sin(x^3)\,dx = \int_0^{\sqrt[3]{\pi}} 2\pi x^2 \sin(x^3)\,dx$$ 3pt

Make $u=x^3$
$du=3x^2dx$

2pt 3pt 3pt

$$=\pi\int_0^{\pi}\frac{2}{3}\sin(u)\,du = \frac{2\pi}{3}\int_0^{\pi}\sin(u)\,du = -\frac{2\pi}{3}\cos(u)\Big|_0^{\pi}$$

3pt

$$=-\frac{2\pi}{3}\Big[\cos(\pi)-\cos(0)\Big] = -\frac{2\pi}{3}[-1-1] = \frac{4}{3}\pi$$

1. b)

Note that $f(x)=g(x)$ for values of x such that
$x+2=x^2$
that is, we need:
$x^2-x-2=0$
then $(x-2)(x+1)=0$

2pt

this is true if $x=-1$ or $x=2$. Also note that $f(x)\geq g(x)$ for $x\in[-1,2]$.

3pt

$$V=\pi\int_{-1}^{2}\left[(x+2)^2-(x^2)^2\right]dx = \pi\int_{-1}^{2}(x+2)^2dx - \pi\int_{-1}^{2}x^4dx$$

Make $u=x+2$
$du=dx$

4pt

$$V=\pi\int_1^4 u^2\,du-\pi\int_{-1}^{2}x^4dx = \pi\frac{u^3}{3}\Big|_1^4 - \pi\frac{x^5}{5}\Big|_{-1}^{2}$$

1.b) cont.

$$V = \pi \frac{x^3}{3}\Big|_1^4 - \pi \frac{x^5}{5}\Big|_{-1}^2 = \pi\left[\frac{(4)^3}{3} - \frac{1}{3}\right] - \pi\left[\frac{(2)^5}{5} - \frac{(-1)^5}{5}\right]$$

$$= \pi\left[\frac{(4)^3}{3} - \frac{1}{3} - \frac{(2)^5}{5} - \frac{1}{5}\right] = \pi\left[\frac{64-1}{3} - \frac{32+1}{5}\right]$$

$$= \pi\left[\frac{63}{3} - \frac{33}{5}\right] = \pi\left(\frac{72}{5}\right)$$ 3pt

2.

4pt

$$f'(x) = 2x - \frac{1}{8x}$$

4pt

$$L = \int_1^e \sqrt{1 + \left(2x - \frac{1}{8x}\right)^2}\,dx = \int_1^e \sqrt{1 + (2x)^2 - 2(2x)\left(\frac{1}{8x}\right) + \left(-\frac{1}{8x}\right)^2}\,dx$$

$$= \int_1^e \sqrt{1 + (2x)^2 - \frac{1}{2} + \left(+\frac{1}{8x}\right)^2}\,dx = \int_1^e \sqrt{(2x)^2 + \frac{1}{2} + \left(\frac{1}{8x}\right)^2}\,dx$$

6pt

$$= \int_1^e \sqrt{(2x)^2 + 2(2x)\left(\frac{1}{8x}\right) + \left(\frac{1}{8x}\right)^2}\,dx = \int_1^e \sqrt{\left(2x + \frac{1}{8x}\right)^2}\,dx$$

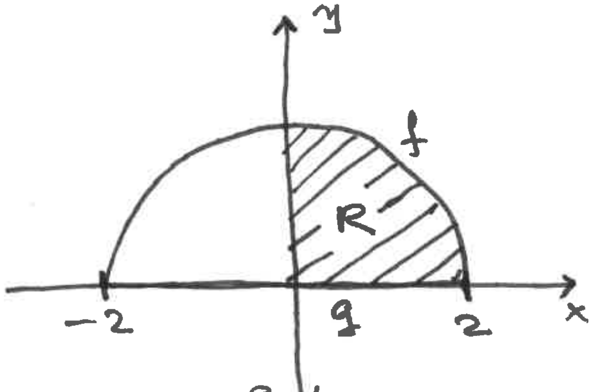
4pt

$$= \int_1^e \left(2x + \frac{1}{8x}\right)dx = \left[x^2 + \frac{1}{8}\ln x\right]_1^e = \left(e^2 + \frac{1}{8}\ln(e)\right) - \left(1^2 + \frac{1}{8}\ln(1)\right)$$

2pt

$$= e^2 + \frac{1}{8} - 1 = e^2 - \frac{7}{8}$$

3. Graph of f and g:



3pt

$$M_y = \int_0^2 x\left[\sqrt{4-x^2} - 0\right] dx$$

$$= \int_0^2 x\sqrt{4-x^2}\, dx$$

Make $u = 4 - x^2$
$du = -2x\,dx$

3pt 2pt

$$= -\frac{1}{2}\int_4^0 u^{1/2}\,du = -\frac{1}{2}\frac{2}{3}u^{3/2}\Big|_4^0 = -\frac{1}{2}\frac{2}{3}\left[0 - 4^{3/2}\right] = \frac{1}{3}(\sqrt{4})^3 = \frac{8}{3}$$

3pt 3pt

$$M_x = \int_0^2 \frac{1}{2}\left[\sqrt{4-x^2}\right]^2 dx = \frac{1}{2}\int_0^2 (4-x^2)\,dx = \frac{1}{2}\left[4x - \frac{x^3}{3}\right]_0^2$$

2pt

$$= \frac{1}{2}\left[4(2) - \frac{(2)^3}{3}\right] = \frac{1}{2}\left[8 - \frac{8}{3}\right] = \frac{1}{2}\left[\frac{16}{3}\right] = \frac{8}{3}$$

2pt

$$A = \int_0^2 \sqrt{4-x^2}\,dx = \frac{1}{4}(\pi \cdot 2^2) = \pi$$ using the formula for the area of a circle.

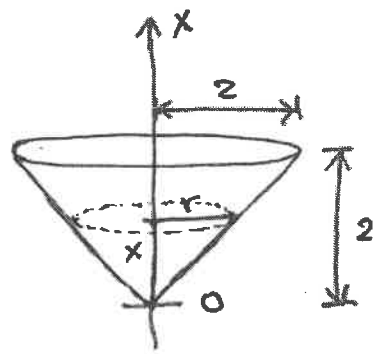
1pt

$$\bar{x} = \frac{M_y}{A} = \frac{\frac{8}{3}}{\pi} = \frac{8}{3\pi}$$

1pt

$$\bar{y} = \frac{M_x}{A} = \frac{8/3}{\pi} = \frac{8}{3\pi}$$

4.



The cross-sections of the tank are circles.

As we can see from the similar triangles, the radius of these cross-sections is equal to x.

8pt

Then $A(x) = \pi x^2$

4pt

$$W = \int_0^2 100(\pi x^2)(4-x)\,dx = 100\pi \int_0^2 x^2(4-x)\,dx$$

5pt

$$= 100\pi \int_0^2 (4x^2 - x^3)\,dx = 100\pi \left[\frac{4}{3}x^3 - \frac{x^4}{4}\right]_0^2$$

$$= 100\pi \left[\frac{4}{3}(2)^3 - \frac{(2)^4}{4}\right] = 100\pi \left[\frac{32}{3} - \frac{16}{4}\right]$$

3pt

$$= \frac{2000}{3}\pi \quad \text{lb}\cdot\text{ft}$$

2pt

5.

a) $\frac{dx}{dt} = 9\cos^2 t(-\sin t)$ $\quad \frac{dy}{dt} = 9\sin^2 t \cos t$

2pt

$$L = \int_0^{\pi/2} \sqrt{(-9\cos^2 t \sin t)^2 + (9\sin^2 t \cos t)^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{81\cos^4 t \sin^2 t + 81\sin^4 t \cos^2 t}\, dt$$

$$= \int_0^{\pi/2} \sqrt{81\cos^2 t \sin^2 t(\cos^2 t + \sin^2 t)}\, dt = \int_0^{\pi/2} \sqrt{81\cos^2 t \sin^2 t}\, dt$$

4pt $\quad$ 2pt $\quad$ 2pt

$$= \int_0^{\pi/2} 9\sqrt{\cos^2 t \sin^2 t}\, dt = 9\int_0^{\pi/2} \cos t \sin t\, dt = 9\int_0^1 u\, du = \frac{9}{2}$$

Make $u = \sin t$

$du = \cos t$

b)

$$L = \int_0^{2\pi} \sqrt{(-9\cos^2 t \sin t)^2 + (9\sin^2 t \cos t)^2}\, dt = \int_0^{2\pi} \sqrt{81\cos^2 t \sin^2 t}\, dt$$

4pt

$$= \int_0^{2\pi} 9\sqrt{\cos^2 t \sin^2 t}\, dt = 9\int_0^{2\pi} |\cos t \sin t|\, dt$$

2pt

$$= 9\left[\int_0^{\pi/2} \cos t \sin t\, dt - \int_{\pi/2}^{\pi} \cos t \sin t\, dt + \int_{\pi}^{3\pi/2} \cos t \sin t\, dt - \int_{3\pi/2}^{2\pi} \cos t \sin t\, dt\right]$$

2pt

$$= 9\left[\int_0^1 u\, du - \int_1^0 u\, du + \int_0^{-1} u\, du - \int_{-1}^0 u\, du\right] = 9\left[\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right] = 18$$

Make $u = \sin t$