1 a)

$$f'(x) = \frac{(9+x^2) - x(2x)}{(9+x^2)^2} = \frac{9-x^2}{(9+x^2)^2} = \frac{(3-x)(3+x)}{(9+x^2)^2}$$

$f'(x) > 0$ if x in $(-3,3)$
$f'(x) < 0$ if x in $(-\infty,-3)$ or $(3,\infty)$

The largest interval containing $x=1$ on which f has an inverse is then $(-3,3)$.

As $f(1) = \frac{1}{10}$, then $(f^{-1})'\left(\frac{1}{10}\right) = \frac{1}{f'(1)} = \frac{(9+(1)^2)^2}{9-(1)^2} = \frac{10^2}{8} = \frac{25}{2}$

b)

$$g(x) = e^{\cos x \ln x}$$

The largest possible domain of g is x in $(0,\infty)$.

$$g'(x) = \frac{d}{dx}\left(e^{\cos x \ln x}\right) = e^{\cos x \ln x}\left[-\sin x \ln x + \cos x \cdot \frac{1}{x}\right]$$

$$= e^{\cos x \ln x}\left[-\sin x \ln x + \frac{\cos x}{x}\right]$$

$$= x^{\cos x}\left[-\sin x \ln x + \frac{\cos x}{x}\right]$$

2.

a) $$\int \frac{dx}{(\sin^{-1}x)\sqrt{1-x^2}} = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\sin^{-1}x| + C$$

$u = \sin^{-1}x$

$du = \frac{1}{\sqrt{1-x^2}}\,dx$

b) $$\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx = -\int_1^0 \frac{du}{1+u^2} = \int_0^1 \frac{du}{1+u^2} = \tan^{-1}(u)\Big|_0^1$$

$u = \cos x$

$du = -\sin x\,dx$

$$= \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

3.

a)

$$\int \frac{\ln y}{\sqrt{y}}\,dy = \int \frac{\ln y}{y^{1/2}}\,dy = \int y^{-1/2} \ln y \, dy$$

$u = \ln y \quad dv = y^{-1/2}\,dy$

$du = \frac{1}{y}\,dy \quad v = 2y^{1/2}$

$$= 2y^{1/2}\ln y - \int \frac{2y^{1/2}}{y}\,dy = 2y^{1/2}\ln y - 2\int y^{-1/2}\,dy =$$

$$= 2y^{1/2}\ln y - 4y^{1/2} + C$$

b)

$$\int_0^{\pi/2} (x^2+1)\sin x\,dx = -(x^2+1)\cos x\Big|_0^{\pi/2} - \int_0^{\pi/2} 2x(-\cos x)\,dx$$

$u = x^2+1 \quad dv = \sin x\,dx$

$du = 2x\,dx \quad v = -\cos x$

$$= -(x^2+1)\cos x\Big|_0^{\pi/2} + 2\int_0^{\pi/2} x\cos x\,dx = -(x^2+1)\cos x\Big|_0^{\pi/2} + 2\left(\cancel{x}\sin x\Big|_0^{\pi/2} - \int_0^{\pi/2} \sin x\,dx\right)$$

$u = x \quad dv = \cos x\,dx$

$du = dx \quad v = \sin x$

$$= -(x^2+1)\cos x\Big|_0^{\pi/2} + 2\left(x\sin x\Big|_0^{\pi/2} + \cos x\Big|_0^{\pi/2}\right)$$

$$= \cancel{-\left(\left(\tfrac{\pi}{2}\right)^2+1\right)\cos\left(\tfrac{\pi}{2}\right)}^{\,0} + (0^2+1)\cos(0) + 2\left(\tfrac{\pi}{2}\sin\left(\tfrac{\pi}{2}\right) - \cancel{0\cdot\sin(0)}^{\,0} + \cancel{\cos\left(\tfrac{\pi}{2}\right)}^{\,0} - \cos(0)\right)$$

$$= \cos(0) + \pi\sin(\pi/2) - 2\cos(0) = 1 + \pi - 2 = \pi - 1$$

4.

L'Hospital's rule

a) $$\lim_{x\to 0^+} \frac{\sin(3x)}{\sin^{-1}(2x)} = \lim_{x\to 0^+} \frac{3\cdot\cos(3x)}{\frac{2}{\sqrt{1-(2x)^2}}} = \lim_{x\to 0^+} \frac{3}{2}\cos(3x)\sqrt{1-4x^2}$$

$$= \frac{3}{2}\cos(0)\sqrt{1} = 3/2$$

b) $$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\frac{1}{x}\ln x} = \lim_{x\to\infty} e^{\frac{\ln x}{x}} = e^{\lim_{x\to\infty}\frac{\ln x}{x}}$$

$$= e^{\lim_{x\to\infty}\frac{\frac{1}{x}}{1}} = e^{\lim_{x\to\infty}\frac{1}{x}} = e^0 = 1$$

by L'Hospital's rule

c) $$\lim_{x\to\infty} \frac{\ln(x+e^{2x})}{x} = \lim_{x\to\infty} \frac{\frac{(1+2e^{2x})}{(x+e^{2x})}}{1} = \lim_{x\to\infty} \frac{(1+2e^{2x})}{x+e^{2x}}$$

L'Hospital's rule

$$= \lim_{x\to\infty} \frac{4e^{2x}}{1+2e^{2x}} = \lim_{x\to\infty} \frac{8e^{2x}}{4e^{2x}} = 2$$

by L'Hospital's rule

L'Hospital's rule.

5.

a)

$$\frac{dy}{dx} = -e^{x+y} = -e^{x} \cdot e^{y}$$

$$e^{-y} dy = -e^{x} dx$$

$$\int e^{-y} dy = -\int e^{x} dx$$

$$-e^{-y} = -e^{x} + c$$

For $x=1$, we want $y=1$:

$-e^{-1} = -e^{1} + c$, then $c = e - e^{-1} = e - \frac{1}{e}$

then the solution is $-e^{-y} = -e^{x} + (e - e^{-1})$

b)

$P(x) = \frac{1}{1+x}$, $Q(x) = 1+x$ because the equation is already in the form $\frac{dy}{dx} + P(x)\, y = Q(x)$.

$S(x) = \int P(x) dx = \int \frac{1}{1+x} dx = \ln|1+x| = \ln(1+x)$ because $x>0$.

Multiply both sides of the equation by $e^{S(x)}$:

$$e^{\ln(1+x)} \frac{dy}{dx} + e^{\ln(1+x)} \frac{1}{1+x} y = e^{\ln(1+x)} (1+x)$$

then (by how $s(x)$ was constructed):

$$\frac{d}{dx}\left(e^{\ln(1+x)} y\right) = e^{\ln(1+x)}(1+x) = (1+x)\cdot(1+x) = (1+x)^2$$

$$\frac{d}{dx}\left((1+x)\, y\right) = (1+x)^2$$

$$(1+x)\, y = \int (1+x)^2 dx = \int u^2 du = \frac{u^3}{3} + c = \frac{(1+x)^3}{3} + c$$

$u = 1+x$
$du = dx$

For $x=1$, we need $y=1$:

$$(1+1)\cdot 1 = \frac{(1+1)^3}{3} + c = \frac{8}{3} + c$$

then $2 = \frac{8}{3} + c$ then $c = 2 - \frac{8}{3} = -\frac{2}{3}$

The solution is:

$$(1+x)\, y = \frac{(1+x)^3}{3} - \frac{2}{3}$$

$$y = \frac{(1+x)^2}{3} - \frac{2}{3(1+x)}$$