1.

3pt

a) $\lim_{x\to\infty}\left(1+\frac{1}{3x}\right)^x = \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{3x}\right)} = e^{\lim_{x\to\infty} x\ln\left(1+\frac{1}{3x}\right)}$

3pt $= e^{\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{3x}\right)}{\frac{1}{x}}} \underset{\text{by l'Hospital's rule}}{=} e^{\lim_{x\to\infty}\frac{\left(1+\frac{1}{3x}\right)^{-1}\left(-\frac{1}{3x^2}\right)}{\left(-\frac{1}{x^2}\right)}} = e^{\lim_{x\to\infty}\frac{1}{3}\left(1+\frac{1}{3x}\right)^{-1}}$

$= e^{1/3}$ 4pt

then $\lim_{n\to\infty}\left(1+\frac{1}{3n}\right)^n = e^{1/3}$

4pt

b) $\lim_{n\to\infty}\left(\frac{5n^2+2n+1}{4-2n-3n^2}\right) = \lim_{n\to\infty}\left(\frac{5+\frac{2}{n}+\frac{1}{n^2}}{\frac{4}{n^2}-\frac{2}{n}-3}\right) = \left(\frac{5+0+0}{0-0-3}\right)$

$= -\frac{5}{3}$ 6pt

a) 4pt

$$\sum_{n=2}^{\infty} \frac{2}{n^2+2n} = \sum_{n=2}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2} \right)$$

$$s_1 = \frac{2}{2^2+2\cdot 2} = \left(\frac{1}{2} - \frac{1}{4} \right)$$

$$s_2 = \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right)$$

$$s_3 = \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{1}{6} \right)$$

$$s_4 = \left(\frac{1}{2} - \cancel{\frac{1}{4}} \right) + \left(\frac{1}{3} - \cancel{\frac{1}{5}} \right) + \left(\cancel{\frac{1}{4}} - \frac{1}{6} \right) + \left(\cancel{\frac{1}{5}} - \frac{1}{7} \right)$$

$$s_j = \left(\frac{1}{2} - \cancel{\frac{1}{4}} \right) + \left(\frac{1}{3} - \cancel{\frac{1}{5}} \right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}} \right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}} \right) + \dots + \left(\cancel{\frac{1}{j-1}} - \cancel{\frac{1}{j+1}} \right)$$

$$+ \left(\cancel{\frac{1}{j}} - \frac{1}{j+2} \right) + \left(\cancel{\frac{1}{j+1}} - \frac{1}{j+3} \right) = \frac{1}{2} + \frac{1}{3} - \frac{1}{j+2} - \frac{1}{j+3}$$ 4pts

then $$\sum_{n=2}^{\infty} \frac{2}{n^2+2n} = \lim_{j \to \infty} s_j = \lim_{j \to \infty} \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{j+2} - \frac{1}{j+3} \right)$$

$$= \frac{1}{2} + \frac{1}{3} - 0 - 0 = \frac{5}{6}.$$ 2pt

2.b)

geometric series.

4pt

$$\sum_{n=1}^{\infty}(-1)^n\left(\frac{2}{3}\right)^{2n} = \sum_{n=1}^{\infty}\left(-\left(\frac{2}{3}\right)^2\right)^n = \sum_{n=1}^{\infty}\left(-\frac{4}{9}\right)^n = \frac{\left(-\frac{4}{9}\right)}{1-\left(-\frac{4}{9}\right)}$$

$$= \frac{-4/9}{1+4/9} = \frac{-4/9}{\frac{13}{9}} = -\frac{4}{13}.$$ 4pt

$\Sigma(-1)^n a_n$

the series is alternating, so:

$$|E_j| < a_{j+1} = \left(\frac{4}{9}\right)^{j+1}$$ 2pt

3.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$.

Limit comparison test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. 2pt

$$\lim_{n\to\infty} \frac{\frac{1}{n\sqrt[n]{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1 > 0.$$ 3pt

then because $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$ diverges too.

1pt

b) $\displaystyle\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^2+1}$

$f(x) = \dfrac{\tan^{-1} x}{x^2+1}$ is decreasing on $[1,\infty)$ and $f(n) = \dfrac{\tan^{-1} n}{n^2+1}$.

$$\int_1^{\infty} \frac{\tan^{-1} x}{x^2+1}\,dx = \lim_{b\to\infty} \int_1^b \frac{\tan^{-1} x}{x^2+1}\,dx \overset{*}{=} \lim_{b\to\infty} \left[\frac{(\tan^{-1} x)^2}{2}\right]_1^b$$ 2pt

$$= \lim_{b\to\infty} \left[\frac{(\tan^{-1} b)^2}{2} - \frac{(\tan^{-1}(1))^2}{2}\right] = \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2\right] < \infty$$ 4pt

then by the integral test, $\displaystyle\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^2+1}$ converges. 1pt

$*$ $\displaystyle\int \frac{\tan^{-1} x}{x^2+1}\,dx = \int u\,du = \frac{u^2}{2} = \frac{(\tan^{-1} x)^2}{2} + C$.

$u = \tan^{-1} x$

$du = \dfrac{1}{x^2+1}\,dx$

3. c)

Limit comparison test with $\sum_{n=2}^{\infty} \frac{1}{n^2}$. 2pt

$$\lim_{n\to\infty} \frac{\frac{1}{n\sqrt{n^2+n-1}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+n-1}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n-1}}$$

4pt

$$= \lim_{n\to\infty} \frac{1}{\frac{1}{n}\sqrt{n^2+n-1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}-\frac{1}{n^2}}} = 1 > 0.$$

1pt

Then because $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges, $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2+n-1}}$ converges.

4. a)

The series $\sum_{n=3}^{\infty} \left| (-1)^{n+1} \frac{1}{(n+1)(n-2)} \right| = \sum_{n=3}^{\infty} \frac{1}{(n+1)(n-2)}$

converges. [for example, by comparison with $\sum_{n=3}^{\infty} \frac{1}{n^2}$: 2pt

$$\lim_{n\to\infty} \frac{\frac{1}{(n+1)(n-2)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{(n+1)(n-2)} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})(1-\frac{2}{n})} = 1 > 0$$ 4pt

Because $\sum_{n=3}^{\infty} \frac{1}{n^2}$ converges then $\sum_{n=3}^{\infty} \frac{1}{(n+1)(n-2)}$ converges]. 1pt

Then the series converges absolutely. 3pt

b) the series $\sum_{n=5}^{\infty} \left| (-1)^{n+1} \left(\frac{n^3+3n+2}{3n^3-2n+1} \right)^n \right| = \sum_{n=5}^{\infty} \left(\frac{n^3+3n+2}{3n^3-2n+1} \right)^n$

converges.

By the root test: 2pt

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n^3+3n+2}{3n^3-2n+1} \right)^n} = \lim_{n\to\infty} \left(\frac{n^3+3n+2}{3n^3-2n+1} \right) = \lim_{n\to\infty} \left(\frac{1+\frac{3}{n^2}+\frac{2}{n^3}}{3-\frac{2}{n^2}+\frac{1}{n^3}} \right)$$

3pt

$$= \frac{1+0+0}{3-0+0} = \frac{1}{3} < 1,$$ then $\sum_{n=5}^{\infty} \left(\frac{n^3+3n+2}{3n^3-2n+1} \right)^n$ converges. 2pt

Then $\sum_{n=5}^{\infty} (-1)^{n+1} \left(\frac{n^3+3n+2}{3n^3-2n+1} \right)^n$ converges absolutely. 3pt

4. c)

the series $\sum_{n=1}^{\infty}\left|(-1)^n n\left(\frac{2}{3}\right)^n\right| = \sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n$ converges.

By the ratio test:

$$\lim_{n\to\infty}\frac{(n+1)\left(\frac{2}{3}\right)^{n+1}}{n\left(\frac{2}{3}\right)^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)\left(\frac{2}{3}\right) = \frac{2}{3} < 1 \text{, then}$$

2pt 3pt

$\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n$ converges 2pt

Then $\sum_{n=1}^{\infty}(-1)^n n\left(\frac{2}{3}\right)^n$ converges absolutely. 3pt

8. a)

By the ratio test: 2pt

$$\lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{(3(n+1))!}x^{n+1}}{\frac{n!}{(3n)!}x^n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{(3n+3)!}}{\frac{n!}{(3n)!}}x\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(3n)!}{n!(3n+3)!}x\right|$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{n!}(n+1)}{\cancel{n!}}\frac{\cancel{(3n)!}}{\cancel{(3n)!}(3n+1)(3n+2)(3n+3)}x\right| = 0 < 1$$ 5pt

Then the power series converges for any value of x.

The interval of convergence is $(-\infty, \infty)$. 3pt

b)

By the ratio test:

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{n+1}}{5^{n+1}(n+9)}}{\frac{(-1)^n x^n}{5^n(n+8)}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}5^n(n+8)x^{n+1}}{(-1)^n 5^{n+1}(n+9)x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1}{5}\frac{(n+8)}{(n+9)}x\right| = \frac{1}{5}|x|$$ 4pt

the power series converges (by the ratio test) if $\frac{1}{5}|x|<1$,

that is, if $|x|<5$. 2pt

The radius of convergence is 5.

For $x=5$: 2pt

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5^n}{5^n(n+8)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+8)},$$ which converges by the alternating series test,

because $\left\{\frac{1}{n+8}\right\}_{n=0}^{\infty}$ is decreasing and $\lim_{n\to\infty} \frac{1}{n+8} = 0$

For $x=-5$: 2pt

$$\sum_{n=0}^{\infty} \frac{(-1)^n(-5)^n}{5^n(n+8)} = \sum_{n=0}^{\infty} \frac{5^n}{5^n(n+8)} = \sum_{n=0}^{\infty} \frac{1}{n+8},$$ which diverges

by limit comparison test with $\sum_{n=0}^{\infty} \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{\frac{1}{n+8}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+8} = 1 > 0$$ and $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges (harmonic series).

Then the interval of convergence of the power series is:

$$(-5, 5]$$