MATH141 QUIZ 8

Name:

1. Use the integral test to show that the following series converges:

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

Hint: You can show that $f(x) = \dfrac{1}{x(\ln x)^2}$ is decreasing on $[2, \infty)$ by showing, instead, that $g(x) = x(\ln x)^2$ is increasing on $[2, \infty)$. For this, you need to show that $g'(x) > 0$.

$g'(x) = (\ln x)^2 + x \cdot 2(\ln x)\frac{1}{x} = (\ln x)^2 + 2(\ln x) > 0$ then $g(x)$ is increasing

(*) then $f(x) = \frac{1}{g(x)}$ is decreasing.

(**) Also $f(n) = \frac{1}{n(\ln n)^2}$.

$$\int \frac{1}{x(\ln x)^2}\,dx = \int \frac{1}{u^2}\,du = \int u^{-2}\,du = -u^{-1} + c = -(\ln x)^{-1} + c$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty}\left[-(\ln x)^{-1}\right]_2^b$$

$$= \lim_{b\to\infty}\left[-(\ln b)^{-1} + (\ln 2)^{-1}\right] = (\ln 2)^{-1} < \infty$$

Then because $\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx < \infty$ and (*) and (**),

the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges by the integral test.

2. Show that the following series converges, using at least two different methods:

$$\sum_{n=1}^{\infty} \frac{2^n}{n^n}$$

Root Test

$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^n}} = \lim_{n\to\infty} \frac{2}{n} = 0 < 1$ then the series converges.

Ratio Test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)^{n+1}}}{\frac{2^n}{n^n}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^{n+1}} \frac{2^{n+1}}{2^n}$$

$$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n(n+1)} 2 = \lim_{n\to\infty} \frac{2}{\frac{(n+1)^n}{n^n}(n+1)} = \lim_{n\to\infty} \frac{2}{\left(\frac{n+1}{n}\right)^n(n+1)}$$

$= \lim_{n\to\infty} \frac{2}{(1+\frac{1}{n})^n(n+1)} = 0$ because we know that $\lim_{n\to\infty} (1+\frac{1}{n})^n = e$.

then the series converge.