1.

a) The characteristic polynomial for the operator can be factored as

$$p(z) = z(z+3)^2((z-2)^2+9)$$

The roots of this polynomial are:

0

-3 with multiplicity two (5pt)

$2 \pm i3$

the general solution to $Ly=0$ is

$$y = c_1 + c_2 e^{-3t} + c_3 t e^{-3t} + c_4 e^{2t}\cos(3t) + c_5 e^{2t}\sin(3t) \quad (4pt)$$

b)

$$W[1-t, e^{-t}] = \det\begin{pmatrix} 1-t & e^{-t} \\ -1 & -e^{-t} \end{pmatrix} = (1-t)(-e^{-t}) - (-1)(e^{-t})$$

$$= (t-1)e^{-t} + e^{-t} = te^{-t} \neq 0 \text{ if } t>0. \quad (4pt)$$

Then $\{1-t, e^{-t}\}$ is a fundamental set and the general solution of the equation is

$$y = c_1(1-t) + c_2 e^{-t} \quad (2pt)$$

c) Write in normal form: $y'' + \dfrac{y}{(x^2-1)(x-3)} = \dfrac{e^x \cos x}{(x^2-1)}$

The coefficients or forcing only fail to be continuous if $x=\pm 1$ or $x=3$. $(1,3)$ is then the largest interval on which the solution is defined given that the initial data is given for $t=2$. (10pt)

2.

$\boxed{\text{Homogeneous problem}}$

$y'' - 16y = 0$

$p(z) = z^2 - 16$ then the roots are 4 and -4.

$y_H(t) = c_1 e^{4t} + c_2 e^{-4t}$ (10pt)

$\boxed{\text{Particular solution}}$

$y'' - 16y = 32e^{-4t}$

$d = 0$

$\mu + i\nu = 4$

$m = 1$

$L(e^{-4t}) = p(-4)\, e^{-4t}$

$[(-4)^2 - 16]e^{-4t} = 0$

$L(te^{-4t}) = p'(-4)e^{-4t} + p(-4)te^{-4t}$

$= [2(-4)]e^{-4t}$ because $p'(z) = 2z$

$= -8e^{-4t}$

then $L(-4te^{-4t}) = 32e^{-4t}$

then $y_p(t) = -4te^{-4t}$ (10pt)

The general solution to the equation is:

$y(t) = y_H(t) + y_p(t) = c_1 e^{4t} + c_2 e^{-4t} - 4te^{-4t}$ (5pt)

Alternatively:

$y'' - 16y = 32e^{-4t}$

$\alpha = 0$

$\mu + i\nu = -4$

$m = 1$

We know that a particular solution has the term:

$y_p(t) = A t e^{-4t}$

then $y_p'(t) = A e^{-4t} - 4A t e^{-4t}$

$y_p''(t) = -\cancel{4}e^{-4t} - 4A e^{-4t} + 16 A t e^{-4t}$

$= -8A e^{-4t} + 16 A t e^{-4t}$

We need

$(-8A e^{-4t} + 16 A t e^{-4t}) - 16(A t e^{-4t}) = 32 e^{-4t}$

i.e. $-8A e^{-4t} = 32 e^{-4t}$

then $A = -4$

A particular solution is $y_p(t) = -4te^{-4t}$ (10 pt)

the general solution to the equation is

$y(t) = y_h(t) + y_p(t) = c_1 e^{4t} + c_2 e^{-4t} - 4te^{-4t}$ (5 pt)

3. Solution to the homogeneous problem

$y'' - 5y' + 6y = 0$

$p(z) = z^2 - 5z + 6 = (z-3)(z-2)$

$y_h(t) = c_1 e^{3t} + c_2 e^{2t}$ (5 pt)

we can split the problem into two:

$\boxed{y_p(I)}$

$y'' - 5y' + 6y = 20\sin(4t)$

$d = 0$

$\mu + i\nu = 4i$

$m = 0$

$$L(e^{4it}) = p(4i)e^{4it}$$
$$= [(4i)^2 - 5(4i) + 6]e^{4it}$$
$$= [-16 - 20i + 6]e^{4it}$$
$$= [-10 - 20i]e^{4it}$$

then
$$L\left(\frac{e^{4it}}{-10-20i}\right) = e^{4it}$$

then
$$L\left(\frac{-2e^{4it}}{1+2i}\right) = 20e^{4it}$$

then
$$y_{pI}(t) = \operatorname{Im}\left(\frac{-2e^{4it}}{1+2i}\right)$$

$$\frac{-2e^{4it}}{1+2i} = \frac{(1-2i)}{(1-2i)}\frac{(-2e^{4it})}{(1+2i)} = \frac{(1-2i)(-2\cos(4t) - 2i\sin(4t))}{1+(2)^2}$$

$$= \frac{-2i\sin(4t) + 4i\cos(4t) - 2\cos(4t) + (2i)(2i\sin(4t))}{5}$$

then $y_{pI}(t) = -\frac{2}{5}\sin(4t) + \frac{4}{5}\cos(4t)$ (10 pt)

$\boxed{y_p \mid II}$ $\quad y'' - 5y + 6y = 36t^2$

forcing:

$d = 2$

$\mu + i\nu = 0$

$m = 0$

$L(1) = L(e^{0\cdot t}) = p(0)\, e^{0t}$

$L(t) = L(t e^{0t}) = p'(0) e^{0t} + p(0)\, t e^{0t}$

$L(t^2) = L(t^2 e^{0t}) = p''(0) e^{0t} + 2p'(0)\, t e^{0t} + p(0)\, t^2 e^{0t}$

then

$L(1) = p(0)$

$L(t) = p'(0) + p(0)\, t$

$L(t^2) = p''(0) + 2p'(0)\, t + p(0)\, t^2$

We have $p(z) = z^2 - 5z + 6$, $p'(z) = 2z - 5$, $p''(z) = 2$

then

$L(1) = 6$

$L(t) = -5 + 6t$

$L(t^2) = 2 - 10t + 6t^2$

then $10L(t) + 6L(t^2) = -50 + \cancel{60t} + 12 - \cancel{60t} + 36t^2 = -38 + 36t^2$

then

$10L(t) + 6L(t^2) + \frac{38}{6} L(1) = 36t^2$

i.e. $L\left(10t + 6t^2 + \frac{38}{6}\right) = 36t^2$, then $y_{pII}(t) = 6t^2 + 10t + \frac{38}{6}$ (7pt)

The general solution to the equation is:

$y(t) = y_h(t) + y_{pI}(t) + y_{pII}(t) = c_1 e^{3t} + c_2 e^{2t} - \frac{2}{5}\sin(4t) + \frac{4}{5}\cos(4t) + 6t^2 + 10t + \frac{38}{6}$

(3pt)

Alternatively:

$\boxed{y_p \mid I}$

$$y'' - 5y' + 6y = 20\sin(4t)$$
$$d = 0$$
$$\mu + iv = 4i$$
$$m = 0$$

y_{p_I} has the form
$$y_{p_I} = A\cos(4t) + B\sin(4t)$$
$$y'_{p_I} = -4A\sin(4t) + 4B\cos(4t)$$
$$y''_{p_I} = -16A\cos(4t) - 16B\sin(4t)$$

We need
$$(-16A\cos(4t) - 16B\sin(4t)) - 5(-4A\sin(4t) + 4B\cos(4t))$$
$$+ 6(A\cos(4t) + B\sin(4t)) = 20\sin(4t)$$

$$[-16A - 20B + 6A]\cos(4t) + [-16B + 20A + 6B]\sin(4t) = 20\sin(4t)$$

then

(i) $-10A - 20B = 0$

(ii) $-10B + 20A = 20$

Adding (i) to (ii) twice: $-50B = 20$, then $B = -\frac{20}{50} = -\frac{2}{5}$

from (i): $-10A - 20\left(-\frac{2}{5}\right) = 0$

then $A = \frac{4}{5}$

(10 pt)

Alternatively

$\boxed{y_p \mid II}$

$$y'' - 5y' + 6y = 36t^2$$

$d = 2$

$\mu + i\nu = 0$

$m = 0$

A particular solution has the form: $y_{pII}(t) = A_0 t^2 + A_1 t + A_2$

then $y'_{pII}(t) = 2A_0 t + A_1$

$y''_{pII}(t) = 2A_0$

We need:

$$(2A_0) - 5(2A_0 t + A_1) + 6(A_0 t^2 + A_1 t + A_2) = 36t^2$$

$$6A_0 t^2 - 10A_0 t + 6A_1 t + 2A_0 - 5A_1 + 6A_2 = 36t^2$$

$$(6A_0)t^2 + (-10A_0 + 6A_1)t + (2A_0 - 5A_1 + 6A_2) = 36t^2$$

then we need to solve the system:

(i) $6A_0 = 36$

(ii) $-10A_0 + 6A_1 = 0$

(iii) $2A_0 - 5A_1 + 6A_2 = 0$

From (i): $A_0 = 6$.

From (ii): $6A_1 = 10A_0 = 10(6) = 60$ then $A_1 = 10$

From (iii):

$6A_2 = 5A_1 - 2A_0 = 5(10) - 2(6) = 50 - 12 = 38$ then $A_2 = \frac{38}{6}$

then $y_{pII}(t) = 6t^2 + 10t + \frac{38}{6}$ (7 pt)

the general solution to the equation is

$y(t) = y_H(t) + y_{pI}(t) + y_{pII}(t) = C_1 e^{3t} + C_2 e^{2t} - \frac{2}{5}\sin(4t) + \frac{4}{5}\cos(4t) + 6t^2 + 10t + \frac{38}{6}$ (3 pt)

4 a)

i) h approaches zero as t goes to infinity (3 pt)

ii) underdamped (conjugate pair of complex roots) (2 pt)

iii)

From the formula for the cosine of the difference of two angles, we see that if $h(t) = Ae^{-5t}\cos(6t-\delta)$ then

$$h(t) = Ae^{-5t}\left[\cos(6t)\cos(\delta) + \sin(6t)\sin(\delta)\right]$$

Comparing the coefficients with the given formula

$$h(t) = e^{-5t}\cos(6t) + e^{-5t}\sin(6t)$$

we see that we need

$$A\cos(\delta) = 1 \qquad A\sin(\delta) = 1$$

then $A^2\cos^2(\delta) + A^2\sin^2(\delta) = 1+1 = 2$

then $A^2 = 2$, then $A = \sqrt{2}$.

we need to find δ such that

$$\cos(\delta) = \frac{1}{\sqrt{2}} \qquad \sin(\delta) = \frac{1}{\sqrt{2}}$$

then $h(t) = \sqrt{2}\, e^{-5t}\cos(6t - \pi/4)$ (7 pt)

b) Need to solve

$$h'' + 16h = 0$$

$$p(z) = z^2 + 16$$

then roots are $\pm i4$

General solution is

$$h(t) = c_1 \cos(4t) + c_2 \sin(4t)$$ (6 pt)

$$h'(t) = -4c_1 \sin(4t) + 4c_2 \cos(4t)$$

We need

$$h(0) = c_1 \cos(0) + c_2 \sin(0) = c_1 = 0$$

$$h'(0) = -4c_1 \sin(0) + 4c_2 \cos(0) = 4c_2 = -4$$

then

$$h(t) = -\sin(4t)$$ (7 pt)