MATH246 Summer II
Exam 3 [100 pt]

Instructions: Number the answer sheets from 1 to 4 and fill out all the information in each of them (sign the Honor Pledge on page 1 only). Solve only one problem in every answer sheet. If you need more space to solve a given problem, use the back of the same answer sheet.
No lecture notes, cheat sheets, books, or electronic devices of any kind are allowed.
For full credit, you need to evaluate any integral you encounter.

1. [25pt] Consider the following initial value problem:

$$y'' + y' - 2y = e^t, \qquad y(0) = 1,\ y'(0) = 1$$

   (a) Determine the Laplace transform $F(s)$ of each side of the equation, taking the initial data into consideration.
   (b) Determine the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial value problem.
   (c) Find the solution to the initial value problem by calculating the inverse Laplace transform of $Y(s)$.

2. [25 pt]

   (a) Show that $\mathbf{x}_1(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}$ and $\mathbf{x}_2(t) = \begin{pmatrix} t^{-1} \\ -t^{-2} \end{pmatrix}$ form a fundamental set of solutions of the system $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & 1 \\ t^{-2} & -t^{-1} \end{pmatrix}\mathbf{x}$ (on the interval $0 < t < \infty$).
   Give a fundamental matrix for this system and calculate the natural fundamental matrix associated to time $t = 1$.
   (b) Consider the matrix $\mathbf{\Phi}(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$. Verify that $\det \mathbf{\Phi}(t) \neq 0$. Find a matrix $A(t)$ such that $\Phi(t)$ solves the problem $\mathbf{\Phi}'(t) = A(t)\mathbf{\Phi}(t)$.
   Hint: multiply by the inverse of $\mathbf{\Phi}(t)$, which exists because $\det \mathbf{\Phi}(t) \neq 0$.
   [5 pt bonus (all or nothing)] Consider the system $\dfrac{d\mathbf{x}}{dt} = A(t)\mathbf{x}$ where $A(t)$ is as in 2b and $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$. What is the second order linear equation that x satisfies?

3. [25pts] The 2×2 matrix A has eigenpairs $\left(2, \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right)$ and $\left(-2, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$.

   (a) Give a fundamental matrix for the system $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$
   (b) Find the solution of the initial value problem: $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.
   (c) Find an invertible matrix V and a diagonal matrix D such that $e^{tA} = Ve^{tD}V^{-1}$ (you do not need to compute e^{tA}).
   (d) Find the matrix A.

4. [25 pt]

   (a) Give a real general solution of the following system: $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\mathbf{x}$
   (b) Give a general solution of the following system: $\dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & -4 \\ 4 & 6 \end{pmatrix}\mathbf{x}$