DEPARTMENT OF MATHEMATICS

Student's Name__________ Course#_____ Prob.#_____ Date_____
Section Instructor__________ Sec.#_____

GRADING

HONOR PLEDGE: I pledge on my honor that I have not given or received any unauthorized assistance on this examination or assignment.
Please write the exact wording of the pledge, followed by your signature, in the space below:

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1.

a) If $\mathcal{L}[y] = Y(s)$, then

$\mathcal{L}[y'] = sY(s) - y(0) \qquad = sY(s) - 1$

$\mathcal{L}[y''] = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s - 1$

We also know: $\mathcal{L}[e^t] = \dfrac{1}{s-1}$

then

$\mathcal{L}[y'' + y' - 2y] = \mathcal{L}[e^t]$

$\mathcal{L}[y''] + \mathcal{L}[y'] - 2\mathcal{L}[y] = \mathcal{L}[e^t]$

$s^2Y(s) - s - 1 + sY(s) - 1 - 2Y(s) = \dfrac{1}{s-1}$

9pt

or $\quad (s^2 + s - 2)Y(s) - s - 2 = \dfrac{1}{s-1}$

b) Then $(s^2+s-2)Y(s) = \dfrac{1}{s-1} + s + 2$

5pt

then $Y(s) = \dfrac{1}{s^2+s-2}\left[\dfrac{1}{s-1} + s+2\right]$

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c)

We can write $Y(s)$ as

11 pt

$$Y(s) = \frac{1}{(s-1)(s+2)}\left[\frac{1}{(s-1)} + (s+2)\right]$$

$$= \frac{1}{(s-1)^2(s+2)} + \frac{(s+2)}{(s-1)(s+2)} = \frac{1}{(s-1)^2(s+2)} + \frac{1}{s-1}$$

For the first term we have the decomposition:

$$\frac{1}{(s-1)^2(s+2)} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{s+2}$$

then

$$1 = A(s-1)(s+2) + B(s+2) + C(s-1)^2$$

If $s=1$, then $1 = 3B$ then $B = \frac{1}{3}$

If $s=-2$, then $1 = 9C$ then $C = \frac{1}{9}$

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Finally, if $s=2$, then

$$1 = 4A + 4B + C = 4A + \frac{4}{3} + \frac{1}{9} = 4A + \frac{12}{9} + \frac{1}{9}$$

then $4A = 1 - \frac{13}{9} = -\frac{4}{9}$ then $A = -\frac{4}{36} = -\frac{1}{9}$

then the solution to the initial value problem is

$$y(t) = \mathcal{L}^{-1}\left[-\frac{1}{9}\frac{1}{(s-1)} + \frac{1}{3}\frac{1}{(s-1)^2} + \frac{1}{9}\frac{1}{(s+2)} + \frac{1}{(s-1)}\right]$$

$$= -\frac{1}{9}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right] + \frac{1}{9}\mathcal{L}^{-1}\left[\frac{1}{(s+2)}\right] + \mathcal{L}^{-1}\left[\frac{1}{s-1}\right]$$

$$= -\frac{1}{9}e^t + \frac{1}{3}te^t + \frac{1}{9}e^{-2t} + e^t$$

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2. a) For $x_1(t)$:

$$\begin{pmatrix} 0 & 1 \\ t^{-2} & -t^{-1} \end{pmatrix}\begin{pmatrix} t \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ t^{-1} - t^{-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} t \\ 1 \end{pmatrix}$$

For $x_2(t)$:

$$\begin{pmatrix} 0 & 1 \\ t^{-2} & -t^{-1} \end{pmatrix}\begin{pmatrix} t^{-1} \\ -t^{-2} \end{pmatrix} = \begin{pmatrix} -t^{-2} \\ t^{-3} + t^{-3} \end{pmatrix} = \begin{pmatrix} -t^{-2} \\ 2t^{-3} \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} t^{-1} \\ -t^{-2} \end{pmatrix}$$ 5pt

A fundamental matrix for this system is

$\Phi(t) = \begin{pmatrix} t & t^{-1} \\ 1 & -t^{-2} \end{pmatrix}$ because $\det \Phi(t) = -t^{-1} - t^{-1} = -2t^{-1} \neq 0$ for $t > 0$

Said otherwise, the Wronskian of $\begin{pmatrix} t \\ 1 \end{pmatrix}, \begin{pmatrix} t^{-1} \\ -t^{-2} \end{pmatrix}$ is not zero for $t > 0$.

5pt

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the natural fundamental matrix associated to time $t=1$ is given by

$$\Phi(t) = \Psi(t)\,\Psi(1)^{-1} =$$

$$\Psi(1) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \text{ then } \Psi(1)^{-1} = \frac{\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}^{T}}{\det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}} = \frac{\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}}{-2}$$

$$= \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

$$\text{then } \Phi(t) = \Psi(t)\,\Psi(1)^{-1} = \begin{pmatrix} t & t^{-1} \\ 1 & -t^{-2} \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} = \begin{pmatrix} 1/2\, t + 1/2\, t^{-1} & \frac{1}{2}t - \frac{1}{2}t^{-1} \\ 1/2 - \frac{1}{2}t^{-2} & \frac{1}{2} + 1/2\, t^{-2} \end{pmatrix}$$

7 pt

b) 8 pt

$$\det \Phi(t) = \cos^2 t - (-\sin^2 t) = \cos^2 t + \sin^2 t = 1$$

$$A(t) = \Phi'(t)\,\Phi(t)^{-1} = \begin{pmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{pmatrix}\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

because

$$\Phi(t)^{-1} = \frac{\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}^T}{\det \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

Bonus points: +5 pt (bonus)

the system reads

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{then} \quad \begin{aligned} x' &= y \quad \text{I} \\ y' &= -x \quad \text{II} \end{aligned}$$

Differentiating I and using II we get that

$$x'' = y' = -x.$$

Then x satisfies $x'' + x = 0$.

3 a)

Two linearly independent solutions are (according to the eigenpairs):

$$x_1(t) = e^{2t}\begin{pmatrix}3\\1\end{pmatrix} \qquad x_2(t) = e^{-2t}\begin{pmatrix}1\\-1\end{pmatrix}$$

then a fundamental matrix is $\Phi(t) = \begin{pmatrix} 3e^{2t} & e^{-2t} \\ e^{2t} & -e^{-2t}\end{pmatrix}$

6pt

b) We need to solve the system:

$$\Phi(0)\, c = \begin{pmatrix}2\\2\end{pmatrix} \qquad \text{for some } c = \begin{pmatrix}c_1\\c_2\end{pmatrix}$$

that is

$$\begin{pmatrix} 3 & 1 \\ 1 & -1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}.$$

By Cramer's rule:

$$c_1 = \frac{\begin{vmatrix} 2 & 1 \\ 2 & -1\end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 1 & -1\end{vmatrix}} = \frac{-2-2}{-3-1} = \frac{-4}{-4} = 1$$

$$c_2 = \frac{\begin{vmatrix} 3 & 2 \\ 1 & 2\end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 1 & -1\end{vmatrix}} = \frac{6-2}{-3-1} = \frac{4}{-4} = -1$$

6pt

c) From a theorem in class we know that

$$e^{tA} = V e^{tD} V^{-1} \quad \text{if} \quad A = VDV^{-1} \text{ with } V \text{ invertible and } D \text{ diagonal.}$$

From a,b we know that

$$A = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$$

then, because

$$\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \frac{\begin{pmatrix} 3 & -1 \\ -1 & -1 \end{pmatrix}^{T}}{\det \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}} = \frac{\begin{pmatrix} 3 & -1 \\ -1 & -1 \end{pmatrix}}{-4} = \begin{pmatrix} -3/4 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix},$$

6pt

d) $$A = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} -3/4 & \frac{1}{4} \\ 1/4 & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -6/4 & \frac{2}{4} \\ -2/4 & -2/4 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{18}{4} - \frac{2}{4} & \frac{6}{4} - \frac{2}{4} \\ -\frac{6}{4} + \frac{2}{4} & \frac{2}{4} + \frac{2}{4} \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ -1 & 1 \end{pmatrix}$$

Then the matrix

$$A = \begin{pmatrix} -4 & 1 \\ -1 & 1 \end{pmatrix}.$$

2pt

4 a)

$$p(z) = \det\begin{pmatrix} z-2 & -1 \\ 1 & z-2 \end{pmatrix} = (z-2)^2+1$$

then eigenvalues are $2 \pm i$

By Cayley-Hamilton: (where $\lambda_1 = 2+i$, $\lambda_2 = 2-i$)

$$(A-\lambda_1 I)(A-\lambda_2 I) = 0$$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} = 0$$

then ~~the~~ one eigenvector associated to $2+i$ is $\begin{pmatrix} i \\ -1 \end{pmatrix}$.

6 pt

Two linearly independent solutions are:

$$x_1 = \mathrm{Re}\left(e^{(2+i)t}\begin{pmatrix} i \\ -1 \end{pmatrix}\right) \qquad x_2 = \mathrm{Im}\left(e^{(2+i)t}\begin{pmatrix} i \\ -1 \end{pmatrix}\right)$$

$$e^{(2+i)t}\begin{pmatrix} i \\ -1 \end{pmatrix} = e^{2t}(\cos(t) + i\sin(t))\begin{pmatrix} i \\ -1 \end{pmatrix}$$

$$= e^{2t}\begin{pmatrix} i\cos(t) - \sin(t) \\ -\cos(t) - i\sin(t) \end{pmatrix} = e^{2t}\begin{pmatrix} -\sin(t) \\ -\cos(t) \end{pmatrix} + i e^{2t}\begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$$

then

$$x_1(t) = e^{2t}\begin{pmatrix} -\sin(t) \\ -\cos(t) \end{pmatrix} \qquad x_2(t) = e^{2t}\begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$$

A general solution is

$$x(t) = c_1 e^{2t}\begin{pmatrix} -\sin(t) \\ -\cos(t) \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}.$$

7pt

b)

$$p(z) = \det\begin{pmatrix} z+2 & 4 \\ -4 & z-6 \end{pmatrix} = (z+2)(z-6)+16$$

$$= z^2-6z+2z-12+16 = z^2-6z+2z+4$$

$$= z^2-4z+4$$

$$= (z-2)^2$$

then the eigenvalue is 2.
then (Cayley-Hamilton):

$$(A-\lambda I)(A-\lambda I) = 0$$

$$\begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} -4 & -4 \\ 4 & 4 \end{pmatrix} = 0$$

the eigenvector associated to 2 is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

One solution to the system is

$$x_1(t) = e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

6 pt

Another solution is given by

$$x_2(t) = e^{\lambda t}\left[w + t(A - \lambda I)w\right]$$

where w is any vector not parallel to the eigenvector associated to the eigenvalue λ.

$$x_2(t) = e^{2t}\left[\begin{pmatrix}1\\0\end{pmatrix} + t\begin{pmatrix}-4 & -4\\4 & 4\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right]$$ is another solution.

$$x_2(t) = e^{2t}\left[\begin{pmatrix}1\\0\end{pmatrix} + t\begin{pmatrix}-4\\4\end{pmatrix}\right] = e^{2t}\left[\begin{pmatrix}1-4t\\4t\end{pmatrix}\right]$$

6 pt