MATH246 Summer II
QUIZ 5

Name:

1. Find the function which has the given Laplace transform

$$Y(s) = \frac{s+1}{s^2 - s - 6}$$

2. Solve the initial value problem using the Laplace transform

$$y'' - y' - 2y = e^{2x}; y(0) = 0, y'(0) = 1$$

Table of Laplace Transforms

$\mathcal{L}[c] = \frac{c}{s}$	$\mathcal{L}[e^{at}\sin(bt)] = \frac{b}{(s-a)^2+b^2}$
$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$	$\mathcal{L}[y'(t)] = s\mathcal{L}[y(t)] - y(0) = sY(s) - y(0)$
$\mathcal{L}[e^{at}] = \frac{1}{s-a}$	$\mathcal{L}[y''(t)] = s^2\mathcal{L}[y(t)] - sy(0) - y'(0) = s^2Y(s) - sy(0) - y'(0)$
$\mathcal{L}[\cos(bt)] = \frac{s}{s^2+b^2}$	$\mathcal{L}[u(t-c)] = \frac{e^{-cs}}{s}$
$\mathcal{L}[\sin(bt)] = \frac{b}{s^2+b^2}$	$\mathcal{L}[u(t-c)j(t-c)] = e^{-cs}\mathcal{L}[j(t)]$
$\mathcal{L}[e^{at}t^n] = \frac{n!}{(s-a)^{n+1}}$	$\mathcal{L}[\sinh(at)] = \frac{a}{s^2-a^2}$
$\mathcal{L}[e^{at}\cos(bt)] = \frac{s-a}{(s-a)^2+b^2}$	$\mathcal{L}[\cosh(at)] = \frac{s}{s^2-a^2}$

Solutions to quiz 5

1.

$$Y(s) = \frac{s+1}{s^2-s-6} = \frac{s+1}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$$

Note $A = \left.\frac{s+1}{s+2}\right|_{s=3} = \frac{3+1}{3+2} = \frac{4}{5}$

$$B = \left.\frac{s+1}{s-3}\right|_{s=-2} = \frac{-2+1}{-2-3} = \frac{-1}{-5} = \frac{1}{5}$$

then $Y(s) = \frac{4}{5}\frac{1}{s-3} + \frac{1}{5}\frac{1}{s+2}$

$$\mathcal{L}^{-1}[Y(s)](t) = \mathcal{L}^{-1}\left[\frac{4}{5}\frac{1}{s-3}\right] + \mathcal{L}^{-1}\left[\frac{1}{5}\frac{1}{s+2}\right]$$

$$= \frac{4}{5}\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] + \frac{1}{5}\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] = \frac{4}{5}e^{3t} + \frac{1}{5}e^{-2t}$$

as can be read from the formulas we derived.

2.

$\mathcal{L}[y](s) = Y(s)$

$\mathcal{L}[y'](s) = sY(s) - y(0) \qquad = sY(s)$

$\mathcal{L}[y''](s) = s^2 Y(s) - s\,y(0) - y'(0) \qquad = s^2 Y(s) - 1$

then $\mathcal{L}[y'' - y' - 2y] = s^2 Y(s) - 1 - sY(s) - 2Y(s)$

$= (s^2 - s - 2)Y(s) - 1$

$\mathcal{L}[e^{2t}](s) = \dfrac{1}{s-2}$

then $(s^2 - s - 2)Y(s) = \dfrac{1}{s-2} + 1$

then $Y(s) = \dfrac{1}{s-2}\dfrac{1}{(s^2-s-2)} + \dfrac{1}{(s^2-s-2)}$

Now $\dfrac{1}{s-2}\dfrac{1}{(s^2-s-2)} = \dfrac{1}{s-2}\dfrac{1}{(s-2)(s+1)} = \dfrac{A}{s-2} + \dfrac{B}{(s-2)^2} + \dfrac{C}{s+1}$

where $B = \left.\dfrac{1}{s+1}\right|_{s=2} = \dfrac{1}{3}$, $\quad C = \left.\dfrac{1}{(s-2)^2}\right|_{s=-1} = \dfrac{1}{9}$

Now if $s=3$, then

$\dfrac{1}{(3-2)}\dfrac{1}{(3-2)(3+1)} = \dfrac{1}{4} = \dfrac{A}{3-2} + \dfrac{B}{(3-2)^2} + \dfrac{C}{3+1} = A + B + \dfrac{C}{4}$

$= A + \dfrac{1}{3} + \dfrac{1}{36}$

then $A = \dfrac{1}{4} - \dfrac{1}{3} - \dfrac{1}{36} = \dfrac{-1}{12} - \dfrac{1}{36} = \dfrac{-4}{36} = -\dfrac{1}{9}$

On the other hand

$$\frac{1}{(s^2-s-2)} = \frac{1}{(s-2)(s+1)} = \frac{D}{s-2} + \frac{E}{s+1}$$

with $D = \left.\frac{1}{s+1}\right|_{s=2} = \frac{1}{3}$, $E = \left.\frac{1}{s-2}\right|_{s=-1} = -\frac{1}{3}$

then

$$Y(s) = -\frac{1}{9}\frac{1}{s-2} + \frac{1}{3}\frac{1}{(s-2)^2} + \frac{1}{9}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2} - \frac{1}{3}\frac{1}{s+1}$$

We can calculate the inverse Laplace transform:

$$y(t) = -\frac{1}{9}e^{2t} + \frac{1}{3}te^{2t} + \frac{1}{9}e^{-t} + \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t}$$

$$= \frac{2}{9}e^{2t} - \frac{2}{9}e^{-t} + \frac{1}{3}te^{2t}.$$