MATH246 Summer II
QUIZ 6

Name:

Solve the initial value problem using the Laplace transform

$$y'' + y' - 2y = te^{2t}; y(0) = 1, y'(0) = 1$$

Hint: show that $\frac{1}{(s-2)^2(s+2)(s-1)} = \frac{A}{s-2} + \frac{B}{(s-2)^2} + \frac{C}{s+2} + \frac{D}{s-1}$, with $A = -\frac{5}{16}$, $B = \frac{1}{4}$, $C = -\frac{1}{48}$, $D = \frac{1}{3}$.

let $Y(s) = \mathcal{L}[y]$

then $\mathcal{L}[y'] = sY(s) - y(0) = sY(s) - 1$

$\mathcal{L}[y''] = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s - 1$

$\mathcal{L}[y'' + y' - 2y] = \mathcal{L}[te^{2t}]$

$= \mathcal{L}[y''] + \mathcal{L}[y'] - 2\mathcal{L}[y] = \mathcal{L}[te^{2t}]$

$(s^2Y(s) - s - 1) + (sY(s) - 1) - 2Y(s) = \frac{1}{(s-2)^2}$ (from the table)

$(s^2 + s - 2)Y(s) - s - 2 = \frac{1}{(s-2)^2}$

then $Y(s) = \frac{1}{(s-2)^2(s^2+s-2)} + \frac{s+2}{(s^2+s-2)}$

$= \frac{1}{(s-2)^2(s+2)(s-1)} + \frac{s+2}{(s+2)(s-1)}$

$= \frac{1}{(s-2)^2(s+2)(s-1)} + \frac{1}{s-1}$

Now:

$$\frac{1}{(s-2)^2(s+2)(s-1)} = \frac{A}{s-2} + \frac{B}{(s-2)^2} + \frac{C}{s+2} + \frac{D}{s-1} \qquad (*)$$

then $B = \left.\frac{1}{(s+2)(s-1)}\right|_{s=2} = \frac{1}{(4)(1)} = \frac{1}{4}$ (multiply (*) by $(s-2)^2$ and make $s=2$)

In the same manner:

$$C = \left.\frac{1}{(s-2)^2(s-1)}\right|_{s=-2} = -\frac{1}{48} \qquad D = \left.\frac{1}{(s-2)^2(s+2)}\right|_{s=1} = \frac{1}{3}$$

Now if $s=3$ (this is an easy value to plug into (*)):

$$\frac{1}{10} = A + B + \frac{C}{5} + \frac{D}{2} = A + \frac{1}{4} - \frac{1}{48\cdot 5} + \frac{1}{6}$$

then $A = \frac{1}{10} - \frac{1}{4} + \frac{1}{48\cdot 5} - \frac{1}{6} = \frac{4\cdot 10}{40} + \frac{1-40}{48\cdot 5} = \frac{-6}{40} + \frac{-39}{6\cdot 40}$

$$= \frac{-36-39}{6\cdot 40} = -\frac{75}{240} = -\frac{15}{48} = -\frac{5}{16}$$

then finally:

$$y(t) = \mathcal{L}^{-1}[Y(s)]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{(s-2)^2(s+2)(s-1)} + \frac{1}{s-1}\right] = \mathcal{L}^{-1}\left[\frac{-5/16}{s-2} + \frac{1/4}{(s-2)^2} + \frac{-1/48}{s+2} + \frac{1/3}{s-1} + \frac{1}{s-1}\right]$$

$$= -5/16\,\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{(s-2)^2}\right] - \frac{1}{48}\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] + \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + \mathcal{L}^{-1}\left[\frac{1}{s-1}\right]$$

$$= -\frac{5}{16}e^{2t} + \frac{1}{4}te^{2t} - \frac{1}{48}e^{-2t} + \frac{1}{3}e^{t} + e^{t}$$