MATH246 Summer II
QUIZ 7

Name:

1. Consider the differential system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

(a) Show that $\begin{pmatrix} 3e^{2t} \\ e^{2t} \end{pmatrix}$ and $\begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix}$ are both solutions for this system.

(b) Give a fundamental matrix for this system and justify your answer.

(c) Give a general solution to this system in vector form.

(d) Compute the natural fundamental matrix for this system associated with $t = 0$.
Hint: The natural fundamental matrix associated with time t_I is given by the formula $\Phi(t) = \Psi(t)\Psi(t_I)^{-1}$, where $\Psi(t)$ is any fundamental matrix for the system.

(e) Solve the initial value problem for this system with $x(0) = -1, y(0) = 1$.

a) For $x_1(t) = \begin{pmatrix} 3e^{2t} \\ e^{2t} \end{pmatrix}$:

$$\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3e^{2t} \\ e^{2t} \end{pmatrix} = \begin{pmatrix} 3e^{2t} + 3e^{2t} \\ 3e^{2t} - e^{2t} \end{pmatrix} = \begin{pmatrix} 6e^{2t} \\ 2e^{2t} \end{pmatrix} = \begin{pmatrix} \frac{d}{dt}(3e^{2t}) \\ \frac{d}{dt}(e^{2t}) \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} 3e^{2t} \\ e^{2t} \end{pmatrix}$$

For $x_2(t) = \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix}$:

$$\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix} = \begin{pmatrix} e^{-2t} - 3e^{-2t} \\ e^{-2t} + e^{-2t} \end{pmatrix} = \begin{pmatrix} -2e^{-2t} \\ 2e^{-2t} \end{pmatrix} = \begin{pmatrix} \frac{d}{dt}(e^{-2t}) \\ \frac{d}{dt}(-e^{-2t}) \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix}$$

Both are solutions.

b) $\Psi(t) = \begin{pmatrix} 3e^{2t} & e^{-2t} \\ e^{2t} & -e^{-2t} \end{pmatrix}$ is a fundamental matrix, because

$$W[x_1, x_2](t) = \det\begin{pmatrix} 3e^{2t} & e^{-2t} \\ e^{2t} & -e^{-2t} \end{pmatrix} = -3e^{0} - 1e^{0} = -4 \neq 0$$

c) A general solution to the system is

$$x(t) = c_1 \begin{pmatrix} 3e^{2t} \\ e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix}$$

d)

$$\Psi(0) = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\Psi(0)^{-1} = \frac{\begin{pmatrix} -1 & -1 \\ -1 & +3 \end{pmatrix}^T}{\det \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}} = \frac{\begin{pmatrix} -1 & -1 \\ -1 & +3 \end{pmatrix}}{-4} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{3}{4} \end{pmatrix}$$

then $\Phi(t) = \Psi(t)\Psi(0)^{-1} = \begin{pmatrix} 3e^{2t} & e^{-2t} \\ e^{2t} & -e^{-2t} \end{pmatrix} \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{3}{4} \end{pmatrix}$

$$= \frac{1}{4} \begin{pmatrix} 3e^{2t} + e^{-2t} & 3e^{2t} - 3e^{-2t} \\ e^{2t} - e^{-2t} & e^{2t} + 3e^{-2t} \end{pmatrix}$$

e)

the solution to the initial value problem is given by:

$$x(t) = \Phi(t) \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3e^{2t} & e^{-2t} \\ e^{2t} & -e^{-2t} \end{pmatrix} \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{3}{4} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3e^{2t} & e^{-2t} \\ e^{2t} & -e^{-2t} \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -e^{-2t} \\ e^{-2t} \end{pmatrix}$$